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Auxiliary matrices for the six-vertex model at $q^N = 1$: II. Bethe roots, complete strings and the Drinfeld polynomial

Christian Korff

School of Mathematics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, UK

E-mail: C.Korff@ed.ac.uk

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Abstract

The spectra of recently constructed auxiliary matrices for the six-vertex model respectively the spin $s = 1/2$ Heisenberg chain at roots of unity are investigated. Two conjectures are formulated both of which are proved for $N = 3$ and are verified numerically for several examples with $N > 3$. The first conjecture identifies an Abelian subset of auxiliary matrices whose eigenvalues are polynomials in the spectral variable. The zeros of these polynomials are shown to fall into two sets. One consists of the solutions to the Bethe ansatz equations which determine the eigenvalues of the six-vertex transfer matrix. The other set of zeros contains the complete strings which encode the information on the degeneracies of the model due to the loop symmetry $\widetilde{sl}_2$ present at roots of 1. The second conjecture then states a polynomial identity which relates the complete string centres to the Bethe roots allowing one to determine the dimension of the degenerate eigenspaces. Its proof for $N = 3$ involves the derivation of a new functional equation for the auxiliary matrices and the six-vertex transfer matrix. Moreover, it is demonstrated in several explicit examples that the complete strings coincide with the classical analogue of the Drinfeld polynomial. The latter is used to classify the finite-dimensional irreducible representations of the loop algebra $\widetilde{sl}_2$. This suggests that the constructed auxiliary matrices not only enable one to solve the six-vertex model but also completely characterize the decomposition of its eigenspaces into irreducible representations of the underlying loop symmetry.

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1. Introduction

This paper is a continuation of a previous work [1] on auxiliary matrices for the six-vertex model respectively the $s = 1/2$ Heisenberg chain at roots of 1,

$$H = \sum_{m=1}^{M} \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \frac{q+q^{-1}}{2}\left(\sigma_m^z \sigma_{m+1}^z - 1\right) \qquad \sigma_{M+1}^{x,y,z} \equiv \sigma_1^{x,y,z}. \tag{1}$$

Here $\sigma_m^{x,y,z}$ denote the respective Pauli matrices acting on the mth site. Throughout this paper the deformation parameter q will be assumed to be a primitive Nth root of unity with $N \geqslant 3$. While the first part [1] of this work was mainly concerned with the construction of the auxiliary matrices and their geometric structure, the present paper focuses on the nature of their spectra. This will allow us to solve the eigenvalue problem of the six-vertex transfer matrix and thus the Hamiltonian (1). In particular, we will derive the Bethe ansatz equations [2–5]

$$\left\{\frac{\sinh\frac{1}{2}\left(u_j^B + \mathrm{i}\gamma\right)}{\sinh\frac{1}{2}\left(u_j^B - \mathrm{i}\gamma\right)}\right\}^M = \prod_{\substack{l=1\\ l\neq j}}^{n_B} \frac{\sinh\frac{1}{2}\left(u_j^B - u_l^B + 2\mathrm{i}\gamma\right)}{\sinh\frac{1}{2}\left(u_j^B - u_l^B - 2\mathrm{i}\gamma\right)} \qquad q = \mathrm{e}^{\mathrm{i}\gamma} \tag{2}$$

from representation theory and identify their solutions as zeros of the auxiliary matrices' eigenvalues. Furthermore, we will demonstrate in concrete examples how the spectra of the auxiliary matrices yield information about the degeneracies connected with the loop symmetry $\widetilde{sl}_2$ present at roots of 1 [6, 7]. Before we describe the results of this paper in detail we give a short introduction into the method of auxiliary matrices and a brief overview of the results obtained in [1].

1.1. Auxiliary matrices from quantum group theory

The concept of auxiliary matrices was originally introduced by Baxter in the context of his solution to the eight-vertex model [8–12] and motivated by the lack of spin-conservation. His approach is described in detail in [13]. While the six-vertex model preserves the total spin, its infinite-dimensional symmetry algebra $\widetilde{sl}_2$ at roots of unity does not. It is for this reason that auxiliary matrices provide the appropriate approach for the discussion of the spectrum and the degenerate eigenspaces. Unlike the coordinate space Bethe ansatz [14] they do not rely on spin-conservation. Also the algebraic Bethe ansatz [15] has serious deficiencies. Away from a root of unity the entries of the monodromy matrix provide a spectrum generating algebra providing a complete set of eigenstates. This ceases to be true when $q^N = 1$ as then certain operator products in the Yang–Baxter algebra vanish. If one wants to resolve the structure of the degenerate eigenspaces at roots of unity the concept of auxiliary matrices is therefore the only method left.

However, as far as the six-vertex model is concerned an explicit construction for an auxiliary matrix has only been given by Baxter for the sectors of vanishing total spin, (cf formula (101) in [10]). His expression applies to generic values of the deformation parameter. More recently, auxiliary matrices related to the six-vertex model for $q^N \neq 1$ have been studied using infinite-dimensional representations of quantum groups [16–19]. In [19] the connection with the lattice model has been explicitly investigated and Baxter's result has been extended to spin sectors different from zero involving formal infinite power series.

Quantum group theory also played the key role in the construction of auxiliary matrices at $q^N = 1$ discussed in [1]. It needs to be emphasized that in contrast to [16–19] the construction in [1] uses finite-dimensional representations which are special to the root-of-unity case.

Furthermore, this approach differs in several essential points from the one outlined by Baxter (see sections 1.2 and 1.3 in [1]). Nevertheless, the key idea remains the same.

Recall that the statistical lattice model is defined in terms of the transfer matrix $T(z)$ (defined in equation (20) below) which besides the deformation parameter q depends on a spectral variable z. The spin-chain Hamiltonian (1) is obtained from the transfer matrix by taking the logarithmic derivative with respect to z and setting $z = 1$ afterwards. One now introduces an additional matrix Q which following Baxter is called 'auxiliary'. Its defining property is the solution of a suitable functional relation which allows one to solve the eigenvalue problem of the transfer matrix T in terms of Q. The main result of [1] was to explicitly solve the following operator functional equation:

$$Q_p(z)T(z) = \phi_1(z)^M Q_{p'}(zq^2) + \phi_2(z)^M Q_{p''}(zq^{-2}). \tag{3}$$

Here ϕ_1, ϕ_2 are scalar functions (cf equation (37) in this paper). The auxiliary matrices Q_p, $Q_{p'}$, $Q_{p''}$ depend on additional complex parameters $p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c} = \mu + \mu^{-1}) \in \mathbb{C}^4$ with $\mathbf{z} \neq 1$, whose appearance is connected with the enhanced symmetry of the six-vertex model at roots of 1. These parameters define points on the following three-dimensional complex hypersurface [20–23]

$$\text{Spec}\, Z: \quad \mathbf{xy} + q^{N'}(\mathbf{z} + \mathbf{z}^{-1}) = \mu^{N'} + \mu^{-N'} \qquad N' = \begin{cases} N & \text{if} \quad N \text{ odd} \\ N/2 & \text{if} \quad N \text{ even.} \end{cases} \tag{4}$$

The points $p' = (\mathbf{x}, q^{N'}\mathbf{y}, q^{N'}\mathbf{z}, \mu q + \mu^{-1}q^{-1})$ and $p'' = (\mathbf{x}, q^{N'}\mathbf{y}, q^{N'}\mathbf{z}, \mu q^{-1} + \mu^{-1}q)$ in the functional equation (3) are determined by representation theory. When N is even one has to make the further restriction $\mathbf{x} = \mathbf{y} = 0$. For details and the explicit definition of the auxiliary matrices we refer the reader to [1]. For a subvariety of Spec$Z$ their definition is given below (see equation (26)) in order to keep this paper self-contained. All matrices in the functional equation (3) have been proved to commute with each other [1] whence it can be written in terms of eigenvalues.

There are two main advantages of considering the auxiliary matrix Q_p as the central object instead of the transfer matrix. First, the Bethe roots u_j^B solving (2) can be directly obtained as zeros of the auxiliary matrices' eigenvalues

$$0 = Q_p\big(z_j^B\big)T\big(z_j^B\big) = \phi_1\big(z_j^B\big)^M Q_{p'}\big(z_j^B q^2\big) + \phi_2\big(z_j^B\big)^M Q_{p''}\big(z_j^B q^{-2}\big) \qquad z_j^B = \mathrm{e}^{u_j^B} q^{-1}. \tag{5}$$

Second and more importantly, Q_p breaks the infinite-dimensional symmetry of the six-vertex model at roots of unity and therefore in general is non-degenerate. Employing the functional equation (3) one can show that this implies that the eigenvalues of the auxiliary matrices must contain factors which are q^2-periodic. Consequently, the eigenvalues of Q_p possess additional zeros besides the Bethe roots which are called complete strings [12, 24]

$$q^N = 1: \quad Q_p\big(z_j^S q^{2\ell}\big) = 0 \qquad \ell = 0, 1, 2, \ldots, N' - 1. \tag{6}$$

The occurrence of these complete strings at finite length M of the spin-chain is characteristic to the root-of-unity case. Note that in contrast to the Bethe roots z_j^B the string centre z^S is not determined via the functional equation (3). This freedom is at the heart of understanding the infinite-dimensional symmetry at roots of unity. As we will see in this paper the structure of the degenerate eigenspaces of the transfer matrix and the Hamiltonian (1) is completely described by the complete strings.

1.2. Results and outline of the paper

The investigation of the spectra of the auxiliary matrices Q_p for arbitrary points on the hypersurface (4) is a quite complicated task since they form a non-Abelian set. That is, for

a generic pair $p_1, p_2 \in \mathrm{Spec} Z$ and any pair of spectral variables $z, w \in \mathbb{C}$ the corresponding auxiliary matrices do in general not commute. Instead one finds [1] that in order to ensure $[Q_{p_1}(z), Q_{p_2}(w)] = 0$ one has to enforce the relations

$$\frac{\mathbf{x}_1 z^{-\varepsilon N'}}{1-\mathbf{z}_1} = \frac{\mathbf{x}_2 w^{-\varepsilon N'}}{1-\mathbf{z}_2} \qquad \frac{\mathbf{y}_1 z^{\varepsilon N'}}{1-\mathbf{z}_1^{-1}} = \frac{\mathbf{y}_2 w^{\varepsilon N'}}{1-\mathbf{z}_2^{-1}} \qquad \varepsilon = 0, 1. \tag{7}$$

Note that this is sufficient to guarantee that all operators in (3) commute. While the non-Abelian character of the auxiliary matrices makes them more powerful as a symmetry it complicates the calculation of the eigenvalues, since the eigenvectors may depend on the additional parameters p, the spectral variable z and the deformation parameter q. We will therefore focus only on an Abelian subset for which the eigenvectors exclusively depend on q and the spectra consist of polynomials in the spectral variable z. This Abelian subset is defined in the following conjecture which we will prove for the case $N = 3$ in this paper.

Conjecture 1. *For integer $N \geqslant 3$ consider the subvariety in the hypersurface (4) defined by*

$$p_\mu = (0, 0, \mu^{-N'}, \mu + \mu^{-1}) \in \mathrm{Spec}\, Z \qquad \mu \in \mathbb{C}^\times. \tag{8}$$

Denote the corresponding auxiliary matrices by $Q_\mu(z) \equiv Q_{p_\mu}(z)$. An explicit definition will be given below, (cf equations (26), (28) and (29)). Then one has the commutation relations

$$[Q_\mu(z), Q_\nu(w)] = 0 \qquad \mu, \nu \in \mathbb{C}^\times \qquad z, w \in \mathbb{C}. \tag{9}$$

Because of this relation we refer to the one-parameter subset of auxiliary matrices Q_μ as 'Abelian'.

For $N = 3$ this conjecture will be shown to be valid by explicitly constructing the intertwiner of the quantum loop algebra $U_q(\widetilde{sl}_2)$ associated with the evaluation representations $\pi_z^{p_\mu}, \pi_w^{p_\nu}$ (see definitions (29), (63) in the text and (A.1) in the appendix). That is, we will demonstrate for $q^3 = 1$ that the tensor products $\pi_z^{p_\mu} \otimes \pi_w^{p_\nu}$ and $\pi_w^{p_\nu} \otimes \pi_z^{p_\mu}$ are isomorphic. Numerical calculations have also been performed for $N = 4, 5, 6, 8$ and the conjecture has been found to be valid.

The conjecture that this assertion holds true for all N is motivated by the observation that the necessary criteria for the existence of such an intertwiner—which are identical with the one shown in (7)—are satisfied. These necessary criteria turn out to be sufficient for the existence when $\mathbf{x}_i, \mathbf{y}_i \neq 0$. The corresponding intertwiners have first been obtained in [25]. When $\mathbf{x}_i = \mathbf{y}_i = 0$ the above criteria are obviously trivially satisfied for any values of z, w and $\mathbf{z}_1, \mathbf{z}_2$. Unfortunately, the parametrization used in [25] does not allow one to take the limit $\mathbf{x}_i, \mathbf{y}_i \to 0$ and to obtain the corresponding intertwiners for the subvariety (8). Thus, we need to prove the existence for this special case which is done by explicit construction in the appendix of this paper for $N = 3$. The case of arbitrary N is left to future work.

It follows from their definition given in equation (26) below that each auxiliary matrix can be decomposed as

$$Q_\mu(z) = \sum_{m=0}^{M} Q_\mu^{(m)} z^m \tag{10}$$

where the coefficients $Q_\mu^{(m)}$ are independent of the spectral variable z. If (9) holds true all the coefficients commute. In fact, one has $\left[Q_\mu^{(m)}, Q_\nu^{(n)}\right] = 0$. Hence, every eigenvalue of the auxiliary matrix $Q_\mu(z)$ can be written in terms of polynomials with the most general form being

$$Q_\mu(z) = \mathcal{N}_\mu z^{n_\infty} P_B(z) P_\mu(z) P_S(z^{N'}, \mu). \tag{11}$$

Here $\mathcal{N}_\mu = \mathcal{N}_\mu(q)$ is a normalization constant not depending on z and the three polynomials are given by

$$P_B(z) = \prod_{j=1}^{n_B} \left(z - z_j^B\right) \tag{12}$$

$$P_\mu(z) = \prod_{j=1}^{n} (z - z_j(\mu)) \tag{13}$$

$$P_S(z^{N'}, \mu) = \prod_{j=1}^{n_S} \prod_{\ell \in \mathbb{Z}_{N'}} \left(z - z_j^S(\mu) q^{2\ell}\right) = \prod_{j=1}^{n_S} \left(z^{N'} - z_j^S(\mu)^{N'}\right). \tag{14}$$

The first polynomial P_B contains the zeros $z_j^B = z_j^B(q)$ of the eigenvalue which do not depend on the parameter μ and for which there is at least one $\ell \in \mathbb{Z}_{N'}$ such that $z_j^B q^{2\ell}$ is not a zero of P_B. We will show below that these zeros are finite solutions of the Bethe ansatz equations (2), whence the notation. Via the polynomial $P_B(z)$ they determine (up to a possible sign factor) the eigenvalue of the six-vertex transfer matrix associated with (11),

$$T(z) = \phi_1(z)^M q^{n_\infty} \frac{P_B(zq^2)}{P_B(z)} + \phi_2(z)^M q^{-n_\infty} \frac{P_B(zq^{-2})}{P_B(z)}. \tag{15}$$

The power n_∞ of the monomial in (11) gives the number of the 'Bethe roots at infinity' (see e.g. [26] and references therein). This expression refers to the parametrization in (2) with $z_j^B = \mathrm{e}^{u_j^B} q^{-1} \to 0$. The appearance of 'infinite' Bethe roots is another feature characteristic to the model at roots of 1. It signals the breakdown of the familiar formula that the number n_B of Bethe roots is related to the number of down spins in the corresponding eigenstate.

The second polynomial P_μ accounts for the possibility of zeros depending on q and μ. They occur because the additional parameters p shift in the functional equation (3). Again we allow only for zeros $z_j(\mu)$ for which there is at least one $\ell \in \mathbb{Z}_{N'}$ such that $z_j(\mu) q^{2\ell}$ is not a zero of P_μ. Using the transformation behaviour of the auxiliary matrices under spin-reversal we will show that

$$P_\mu(z) = \prod_{j=1}^{n_B} \left(z - z_j^B \mu^2\right) = \mu^{2n_B} P_B(z\mu^{-2}). \tag{16}$$

The last factor P_S contains the contribution of the complete N'-strings (6), where the string centre z_j^S may or may not depend on μ. The additional dependence on the deformation parameter q is suppressed in the notation. As mentioned before the contribution of the complete strings encodes the information on the degeneracies connected with the loop symmetry at roots of unity. In particular, the number n_S of complete strings is related to the dimension of the corresponding degenerate eigenspace of the transfer matrix. In order to arrive at this result we need to determine the number of possible eigenvalues (11) in a degenerate eigenspace of the transfer matrix.

In the case $N = 3$ this information will be derived from the following functional equation:

$$N = 3: \quad Q_\mu(z) Q_\nu(z\nu^2 q^2) = Q_{\mu\nu q}(z\nu^2 q^2)[q^M (z-1)^M T(zq) + (zq^2 - 1)^M]. \tag{17}$$

The above identity relates a product of two complete string contributions (14) to a single one, thus imposing severe restrictions on the possible μ-dependence of the string centres $z_j^S(\mu)$.

One finds that only two possibilities are allowed: namely, one has

$$\text{either} \quad z_j^S(\mu) = z_j^S \qquad \text{or} \qquad z_j^S(\mu) = z_j^S \mu^2. \tag{18}$$

Here z_j^S denotes the (constant) value of the string centres in the limit $\mu \to 1$. These values are fixed in terms of the Bethe roots via the following remarkable identity which is also deduced from (17),

$$\lim_{\mu\to 1} \mathcal{N}_\mu P_S(z^{N'}, \mu) = z^{-n_\infty} \sum_{\ell\in\mathbb{Z}_{N'}} \frac{q^{2(\ell+1)n_\infty}(zq^{2\ell}-1)^M}{P_B(zq^{2\ell})P_B(zq^{2(\ell+2)})}. \tag{19}$$

Both results taken together imply that for each eigenvalue (15) of the transfer matrix which possesses a fixed number of Bethe roots, finite and infinite ones, there are 2^{n_S} possible eigenvalues of the auxiliary matrix. Since the auxiliary matrices break the infinite-dimensional symmetry of the six-vertex model as well as spin-reversal symmetry they are non-degenerate. Thus, the corresponding 2^{n_S} eigenstates yield a basis for the degenerate eigenspace of the transfer matrix.

Note that the crucial functional equation (17) is only valid for $N = 3$ and must be modified for $N > 3$. Nevertheless, the outcome ought to hold true in general leading to the formulation of the second conjecture.

Conjecture 2. *The identity (19) and the restriction (18) on the μ-dependence of the complete string centres not only holds true for $N = 3$ but applies to all primitive roots of unity of order $N \geqslant 3$. This in particular implies that each eigenvalue of the six-vertex transfer matrix allows for 2^{n_S} eigenvalues of the auxiliary matrix.*

While we will provide a proof of this identity only for $N = 3$ we performed numerical checks for $N = 5, 6, 8$ verifying the second conjecture also in these cases. Note also that this assertion coincides with previous results in the literature [6, 24, 28, 27] obtained by numercial calculations and use of the loop algebra symmetry $\widetilde{sl}_2$ which has been established in the commensurate sectors $2S^z = 0 \bmod N$ [6, 7]. In this context Fabricius and McCoy suggested in [28] an expression for the classical analogue of the Drinfeld polynomial [29]. The latter describes the finite-dimensional irreducible representations of the loop algebra [30].

The main result of this paper is that the contribution of the complete strings (19) of the auxiliary matrices constructed in [1] coincides with the proposed expression for the classical analogue of the Drinfeld polynomial. We will address this point in the last section of this paper when discussing concrete examples. They suggest that the spectra of the auxiliary matrices describe the decomposition of the eigenspaces into irreducible representations of the loop algebra. This is of particular importance as the auxiliary matrices have been defined for all spin sectors, while the loop algebra generators have only been constructed for the sectors where the total spin is a multiple of the order N.

The outline of the paper is as follows. In section 2 we introduce our conventions in defining the six-vertex model and review the definition and properties of the auxiliary matrices. Section 3 discusses the implications of the functional equation (3) for the form of the eigenvalues (11). Section 4 deals with the transformation under spin-reversal, a symmetry which is broken by the auxiliary matrices. Section 5 shows that the eigenvalues of the auxiliary matrices occur always in pairs of opposite momenta and gives the relation between them. Section 6 contains the proof of the crucial functional equation (17). Section 7 summarizes the results. In particular, the connection with the representation theory of the loop algebra $\widetilde{sl}_2$ is made.

2. Definitions

In order to keep this paper self-contained we briefly recall our conventions for the definition of the six-vertex model. The transfer matrix is given as the following trace over an operator product:

$$T(z) = \operatorname*{tr}_0 R_{0M}(z) R_{0M-1}(z) \dots R_{01}(z) \tag{20}$$

with

$$R = \frac{a+b}{2} 1 \otimes 1 + \frac{a-b}{2} \sigma^z \otimes \sigma^z + c\sigma^+ \otimes \sigma^- + c'\sigma^- \otimes \sigma^+ \qquad \sigma^\pm = \frac{\sigma^x \pm \mathrm{i}\sigma^y}{2} \tag{21}$$

being defined over $\mathbb{C}^2 \otimes \mathbb{C}^2$ in terms of the Boltzmann weights of the six-allowed vertex configurations,

$$a = \rho \qquad b = \rho \frac{(1-z)\,q}{1-zq^2} \qquad c = \rho \frac{1-q^2}{1-zq^2} \qquad c' = cz. \tag{22}$$

For convenience we shall henceforth set the arbitrary normalization factor to $\rho \equiv 1$. The lower indices in (20) indicate on which pair of spaces the R-matrix acts in the $(M+1)$-fold tensor product of $\mathbb{C}^2$. The explicit dependence on the parameter q is suppressed in the notation.

The six-vertex transfer matrix possesses a number of finite symmetries given by the vanishing of the following commutators:

$$[T(z), S^z] = [T(z), \mathfrak{R}] = [T(z), \mathfrak{S}] = 0 \tag{23}$$

where the respective operators are defined as follows:

$$S^z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z \qquad \mathfrak{R} = \sigma^x \otimes \dots \otimes \sigma^x \qquad \mathfrak{S} = \sigma^z \otimes \dots \otimes \sigma^z = (-1)^{M/2-|S^z|}. \tag{24}$$

The first operator is the total spin, the second invokes spin-reversal and the third has eigenvalue $+1$ or -1 depending whether the number of down spins n in a state is even or odd. The finite symmetries and the properties of the Boltzmann weights (1) can be used to derive for spin-chains of even length, $M = 2M'$, the following useful relations of the transfer matrix:

$$\begin{aligned} &T(z, q^{-1}) = T(z^{-1}, q) \\ &T(z, -q) = \mathfrak{S} T(z, q) = T(z, q)\mathfrak{S} \\ &T(zq^{-2}, q) = b(z^{-1})^{-M} T(z^{-1}, q)^t. \end{aligned} \tag{25}$$

Here we have temporarily introduced the explicit dependence on the deformation parameter q in the notation. The (25) transformation properties impose restrictions on the spectrum of the transfer matrix. Henceforth, we shall assume M to be even.

2.1. A one-parameter family of auxiliary matrices

As explained in the introduction only a subclass of the auxiliary matrices constructed in [1] will be considered, namely the ones associated with nilpotent representations. In terms of the hypersurface (4) this class of auxiliary matrices corresponds to the points (8). We now explicitly define this one-parameter family of auxiliary matrices setting

$$Q_\mu(z) = \operatorname*{tr}_0 L^\mu_{0M}(z/\mu) L^\mu_{0M-1}(z/\mu) \cdots L^\mu_{01}(z/\mu) \qquad L^\mu_{0m} \in \mathrm{End}(V_0 \otimes V_m) \qquad \mu \in \mathbb{C}^\times. \tag{26}$$

Here the operators in the trace can be expressed as 2×2 matrices with operator-valued entries

$$L^\mu = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix} = A_\mu \otimes \sigma^+\sigma^- + B_\mu \otimes \sigma^+ + C_\mu \otimes \sigma^- + D_\mu \otimes \sigma^-\sigma^+. \tag{27}$$

The operators $A_\mu, B_\mu, C_\mu, D_\mu \in \mathrm{End}(\mathbb{C}^{N'})$ are given as

$$\begin{aligned} A_\mu(w) &= wq\pi^\mu(t) - \pi^\mu(t)^{-1} \\ B_\mu(w) &= wq(q-q^{-1})\pi^\mu(t)\pi^\mu(f) \\ C_\mu &= (q-q^{-1})\pi^\mu(e)\pi^\mu(t)^{-1} \\ D_\mu(w) &= wq\pi^\mu(t)^{-1} - \rho_-\pi^\mu(t) \end{aligned} \tag{28}$$

with the $N' \times N'$ matrices $\pi^\mu(t), \pi^\mu(e), \pi^\mu(f)$ defined through the following action on the standard basis $\{v_n\}$ in $\mathbb{C}^{N'}$ [20, 21][1]:

$$\begin{aligned} &\pi^\mu(t)^2 v_n = \mu^{-1}q^{-2n-1}v_n \qquad \pi^\mu(f)v_n = v_{n+1} \qquad \pi^\mu(f)v_{N'-1} = 0 \\ &\pi^\mu(e)v_n = \frac{\mu + \mu^{-1} - \mu q^{2n} - \mu^{-1}q^{-2n}}{(q-q^{-1})^2}v_{n-1}. \end{aligned} \tag{29}$$

The matrices $\pi^\mu(t)^2, \pi^\mu(e), \pi^\mu(f)$ define a representation π^μ of the quantum group $U_q(sl_2)$ at $q^N = 1$, i.e. they are subject to the relations

$$\begin{aligned} &\pi^\mu(t)\pi^\mu(e)\pi^\mu(t)^{-1} = q\pi^\mu(e) \\ &\pi^\mu(t)\pi^\mu(f)\pi^\mu(t)^{-1} = q^{-1}\pi^\mu(t) \\ &[\pi^\mu(e), \pi^\mu(f)] = \frac{\pi^\mu(t)^2 - \pi^\mu(t)^{-2}}{q-q^{-1}}. \end{aligned} \tag{30}$$

The elements generating the centre of the quantum group take the values

$$\pi^\mu(f)^{N'} = \pi^\mu(e)^{N'} = 0 \qquad \pi^\mu(t)^{2N'} = \mu^{-N'} \tag{31}$$

and

$$q\pi^\mu(t)^2 + q^{-1}\pi^\mu(t)^{-2} + (q-q^{-1})^2\pi^\mu(f)\pi^\mu(e) = \mu + \mu^{-1}. \tag{32}$$

There are several advantages of defining the auxiliary matrix in terms of the representation π^μ. As a consequence of the quantum group relations one has the identity

$$L^\mu_{12}(w/z)L^\mu_{13}(w)R_{23}(z) = R_{23}(z)L^\mu_{13}(w)L^\mu_{12}(w/z) \tag{33}$$

which implies that the auxiliary matrix and the transfer matrix commute,

$$[Q_\mu(w), T(z)] = 0. \tag{34}$$

In addition, one derives from the following non-split exact sequence of evaluation representations π^μ_w[1]

$$0 \to \pi^{\mu q}_{w'} \hookrightarrow \pi^\mu_w \otimes \pi^0_z \to \pi^{\mu q^{-1}}_{w''} \to 0 \qquad w = w'q^{-1} = w''q = z/\mu \tag{35}$$

the specialization of the functional equation (3) to the subvariety (8)

$$Q_\mu(z)T(z) = \phi_1(z)^M Q_{\mu q}(zq^2) + \phi_2(z)^M Q_{\mu q^{-1}}(zq^{-2}). \tag{36}$$

Here π^0_z is the root-of-unity limit of the two-dimensional fundamental representation defining the six-vertex model. The corresponding point on $\mathrm{Spec}Z$ is $p^0 = (0, 0, q^{N'}, q^2 + q^{-2})$. The complex numbers w, z play the role of evaluation parameters respectively spectral variables. The coefficient functions are given by[2]

$$\phi_1(z,q) = b(z,q)q^{-\frac{1}{2}} \qquad \text{and} \qquad \phi_2(z) = q^{\frac{1}{2}}. \tag{37}$$

[1] This parametrization of the auxiliary matrices and the root-of-unity representation slightly differs from the one used in [20, 21] respectively [1]. Instead of using the parameter λ (cf equation (43), section 2 in [1]) it is more favourable to use the variable $\mu = \lambda^{-1}q^{-1}$ as this facilitates the identification when taking the nilpotent limit from a generic cyclic representation $p = \varphi(\xi, \zeta, \lambda)$.

[2] These coefficients are obtained from the one in [1] by setting $\rho_\pm(z) = (zq)^{\frac{1\pm1}{2}}$ in equation (69) (section 3).

All matrices in equation (36) have been shown to commute, whence they can be simultaneously diagonalized and the eigenvalues of T can be expressed in terms of the eigenvalues of the respective auxiliary matrices.

For $N = 3$ we will show that the auxiliary matrices obey the stronger commutation relations (9). For $N > 3$ we assume that conjecture 1 holds for the reasons stated in the introduction.

Recall from [1] that the auxiliary matrices for generic $p \in \mathrm{Spec}Z$ break all the finite symmetries of the six-vertex model. However, the one-parameter family (26) which constitutes a subvariety preserves two of the finite symmetries of the six-vertex transfer matrix, namely

$$[Q_\mu(z), S^z] = [Q_\mu(z), \mathfrak{S}] = 0. \tag{38}$$

Spin-reversal symmetry on the other hand remains broken [1],

$$\mathfrak{R} Q_\mu(z) \mathfrak{R} = Q_{\mu^{-1}}(z\mu^{-2}) = (-zq/\mu)^M Q_\mu(z^{-1}q^{-2}\mu^2)^t. \tag{39}$$

Employing the $U_q(sl_2)$ algebra automorphism $e \to f$, $f \to e$, $t^{\pm 1} \to t^{\pm 1}$, $q \to q^{-1}$ one proves the additional relation

$$Q_\mu(z, q) = Q_\mu(zq^2, q^{-1})^t \tag{40}$$

which allows one to derive the adjoint of the auxiliary matrix

$$Q_\mu(z, q)^* = Q_{\bar{\mu}}(\bar{z}, q^{-1})^t = Q_{\bar{\mu}}(\bar{z}q^{-2}, q). \tag{41}$$

These identities will prove crucial in the following investigation of the eigenvalues (11).

3. The TQ equation in terms of eigenvalues

We start our analysis of the spectra of the auxiliary matrices by inserting the expression (11) in the functional equation (36). Recall from the introduction that (11) is the most general form of the eigenvalues provided (9) is true. By abuse of notation we will denote the operators and their eigenvalues by the same symbols. We obtain from (36),

$$\begin{aligned} T(z)\mathcal{N}_\mu P_B(z) P_\mu(z) P_S(z^{N'}, \mu) = {} & \phi_1(z)^M \mathcal{N}_{\mu q} q^{2n_\infty} P_B(zq^2) P_{\mu q}(zq^2) P_S(z^{N'}, \mu q) \\ & + \phi_2(z)^M \mathcal{N}_{\mu q^{-1}} P_B(zq^{-2}) P_{\mu q^{-1}}(zq^{-2}) P_S(z^{N'}, \mu q^{-1}). \end{aligned}$$

The eigenvalues of the transfer matrix (20) are independent of μ, whence the polynomials P_μ, $P_{\mu q}$, $P_{\mu q^{-1}}$ must cancel on both sides of the functional equation (36) except for a constant factor $q^{\pm 2n}$. Up to a possible renumeration of the zeros $z_j(\mu)$, this implies the following relation:

$$z_j(\mu q) = z_j(\mu) q^2. \tag{42}$$

By the same argument one deduces that the complete string centres $z_j^S(\mu)$ can only differ by powers of q for $\mu \to \mu q^{\pm 1}$,

$$P_S(z^{N'}, \mu) = P_S(z^{N'}, \mu q) \quad \Rightarrow \quad P_S(z^{N'}, \mu) = P_S(z^{N'}, \mu^N). \tag{43}$$

In addition, we can conclude that the ratios $\mathcal{N}_{\mu q}/\mathcal{N}_\mu$, $\mathcal{N}_{\mu q^{-1}}/\mathcal{N}_\mu$ of the normalization factors are independent of μ,

$$\mathcal{N}_{\mu q}/\mathcal{N}_\mu = \mathcal{N}_\mu/\mathcal{N}_{\mu q^{-1}}. \tag{44}$$

Thus, we deduce the following preliminary form of the eigenvalues of the transfer matrix:

$$T(z) = \phi_1(z)^M q^{2n_\infty + 2n_B} \frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu} \frac{P_B(zq^2)}{P_B(z)} + \phi_2(z)^M q^{-2n_\infty - 2n_B} \frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu} \frac{P_B(zq^{-2})}{P_B(z)}. \tag{45}$$

This form is 'preliminary' as we will determine the ratios of the normalization constants below. Evaluating the left-hand side of the Q relation at a zero $z = z_j^B$ we obtain also a preliminary form of the Bethe ansatz equations,

$$0 = \phi_1\big(z_j^B\big)^M \mathcal{N}_{\mu q} q^{2n_\infty} P_B\big(z_j^B q^2\big) + \phi_2\big(z_j^B\big)^M q^{-2n_\infty} \mathcal{N}_{\mu q^{-1}} P_B\big(z_j^B q^{-2}\big). \tag{46}$$

These equations ensure that the eigenvalues of the transfer matrix have residue zero when the limit $z \to z_j^B$ is taken. In fact, one finds that

$$\lim_{z\to z_j^B} T(z) = \lim_{z\to z_j^B} \Bigg(\phi_1(z)^M q^{2n_\infty+2n_B} \frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu} \frac{P_B(zq^2)P_B(zq^{-2})}{P_B(zq^{-2})P_B(z)} + \phi_2(z)^M q^{-2n_\infty-2n_B} \frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu} \frac{P_B(zq^{-2})}{P_B(z)} \Bigg) = 0.$$

Below we will see that the zeros of the polynomial P_B in fact coincide with the finite solutions of the Bethe ansatz equations (2), i.e. they are the Bethe roots at $q^N = 1$. Note that it might also happen that the zeros $z_j^B = 1$ and $z_j^B = q^{-2}$ simultaneously occur. While this possibility looks problematic in light of the parametrization used in (2), equation (46) shows that it does not pose a problem as both sides of the equation then vanish. Also the limit $z \to 1$ yielding the momentum eigenvalue in (45) stays well defined. We therefore include these zeros in the set of Bethe roots.

4. Transformation under spin-reversal

We now discuss the implications of spin-reversal. First note that we can deduce from (34), (9) and the integrability of the six-vertex model, $[T(z), T(w)] = 0$, that the auxiliary and transfer matrix have common eigenvectors which neither depend on the spectral variable z nor on the parameter μ. Let $v = v(q)$ be such a common eigenvector with eigenvalue (11). Then we find according to (39) that

$$\begin{aligned} Q_\mu(z)\mathfrak{R}v &= \mathfrak{R}Q_{\mu^{-1}}(z\mu^{-2})v \\ &= \mathcal{N}_{\mu^{-1}} z^{n_\infty} \mu^{-2n_\infty} P_B(z\mu^{-2}) P_{\mu^{-1}}(z\mu^{-2}) P_S(z^{N'}\mu^{-2N'}, \mu^{-1})\mathfrak{R}v. \end{aligned} \tag{47}$$

Furthermore, the above eigenvalue must satisfy the functional relation (36),

$$\begin{aligned} Q_\mu(z)T(z)\mathfrak{R}v &= [\phi_1(z)^M Q_{\mu q}(zq^2) + \phi_2(z)^M Q_{\mu q^{-1}}(zq^{-2})]\mathfrak{R}v \\ \mathfrak{R}Q_{\mu^{-1}}(z\mu^{-2})T(z)v &= \mathfrak{R}[\phi_1(z)^M Q_{\mu^{-1}q^{-1}}(z\mu^{-2}) + \phi_2(z)^M Q_{\mu^{-1}q}(z\mu^{-2})]v. \end{aligned} \tag{48}$$

Here we have exploited that the transfer matrix is invariant under spin-reversal. In the previous section, we verified that the eigenvalues of $T(z)$ have $3n_B$ zeros at $z_j^B, z_j^B q^{\pm 2}$. These zeros correspond to the finite Bethe roots as we will see shortly. In equation (48) these zeros must originate from the polynomial

$$P_{\mu^{-1}}(z\mu^{-2}) = \mu^{-2n} \prod_{j=1}^{n} (z - z_j(\mu^{-1})\mu^2)$$

as the factors $P_B(z\mu^{-2})$ and the contribution of the complete strings $P_S(z^{N'}\mu^{-2N'}, \mu^{-N})$ cancel on both sides of the equality sign. In fact, replacing $\mu \to \mu^{-1}$ we obtain the expression

$$T(z) = \phi_1(z)^M \frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu} \frac{P_{\mu q^{-1}}(z\mu^2)}{P_\mu(z\mu^2)} + \phi_2(z)^M \frac{\mathcal{N}_{\mu q}}{\mathcal{N}_\mu} \frac{P_{\mu q}(z\mu^2)}{P_\mu(z\mu^2)}.$$

We saw already earlier that $T(z)$ does not contain any poles, i.e. the zeros $z_j(\mu)\mu^{-2}$ have now to satisfy the preliminary Bethe ansatz equations,

$$0 = \phi_1(z_j(\mu)\mu^{-2})^M \mathcal{N}_{\mu q^{-1}} P_{\mu q^{-1}}(z_j(\mu)) + \phi_2(z_j(\mu)\mu^{-2})^M \mathcal{N}_{\mu q} P_{\mu q}(z_j(\mu)).$$

Consequently, we have the zeros $z_j(\mu)\mu^{-2}, z_j(\mu)\mu^{-2}q^{\pm 2}$ of the eigenvalue which must coincide with the zeros $z_j^B, z_j^B q^{\pm 2}$. Hence, after a possible renumeration we are led to the conclusion

$$z_j^B = z_j(\mu)\mu^{-2} \tag{49}$$

which proves relation (16). Using this identification we can now determine the ratios of the normalization constants by comparing with the earlier expression (45) for the transfer matrix eigenvalue and employing (44),

$$\left(\frac{\mathcal{N}_{\mu q^{-1}}}{\mathcal{N}_\mu}\right)^2 = \left(\frac{\mathcal{N}_\mu}{\mathcal{N}_{\mu q}}\right)^2 = q^{2n_\infty + 4n_B} \Rightarrow \mathcal{N}_{\mu q} = \pm \mathcal{N}_\mu q^{-n_\infty - 2n_B}. \tag{50}$$

Inserting this result back into (45) the final expression (15) for the six-vertex transfer matrix eigenvalue associated with (11) is obtained up to a possible sign factor. This ambiguity is due to the square root in (50).

The missing step in order to compare this result with the outcome of the coordinate space Bethe ansatz is to verify that the equations (46) coincide with (2). Employing (44) a simple calculation gives

$$\left(\frac{1 - z_j^B q^2}{q - z_j^B q}\right)^M = q^{2n_\infty + 2n_B - M} \prod_{\substack{k=1 \\ k \neq j}}^{n_B} \frac{z_j^B q^2 - z_k^B}{z_j^B - q^2 z_k^B}. \tag{51}$$

Except for the additional phase in front of the product this coincides with (2) if we identify $z = \mathrm{e}^u q^{-1}, q = \mathrm{e}^{\mathrm{i}\gamma}$. For real eigenvectors we will show momentarily that the phase factor is equal to one. In the general case of complex eigenvectors we do not have a proof that the phase factor is always trivial. However, numerical calculations for $N = 3, 4, 5, 6$ and spin chains up to the length $M = 11$ have so far not produced any counter example.

5. Pairs of eigenvalues

In this section we are going to exploit the relations (39) and (40) of the auxiliary matrix to show that the eigenvalues occur in pairs. Combining these two identities we obtain

$$Q_\mu(z, q) = (-zq/\mu)^M Q_{\mu^{-1}}(z^{-1}q^{-2}, q)^t = (-zq/\mu)^M Q_{\mu^{-1}}(z^{-1}, q^{-1}). \tag{52}$$

According to (9) we can find eigenvectors v which only depend on the deformation parameter, i.e. $v = v(q)$. Equation (52) then shows that the eigenvectors come either in pairs, $(v(q), v(q^{-1}))$ with

$$Q_\mu(z, q)v(q) = \mathcal{N}_\mu(q) z^{n_\infty} P_B(z, q) P_\mu(z, q) P_S(z^{N'}, \mu^N) v(q) \tag{53}$$

and

$$\begin{aligned} Q_\mu(z, q)v(q^{-1}) = (-zq/\mu)^M \mathcal{N}_{\mu^{-1}}(q^{-1}) z^{-n_\infty} P_B(z^{-1}, q^{-1}) P_{\mu^{-1}} \\ \times (z^{-1}, q^{-1}) P_S(z^{-N'}, \mu^{-N}) v(q^{-1}) \end{aligned}$$

or are real, $v(q) = v(q^{-1}) = \overline{v(q)}$, in case of which the two eigenvalues above must be equal. From (25) we infer that the two eigenvectors $(v(q), v(q^{-1}))$ have momentum k of opposite sign with

$$\mathrm{e}^{\mathrm{i}k} := \lim_{z \to 1} T(z, q) = \pm q^{\frac{M}{2} - n_\infty - 2n_B} \prod_{j=1}^{n_B} \frac{1 - z_j^B(q) q^2}{1 - z_j^B(q)}.$$

Rewriting the eigenvalue of $v(q^{-1})$ in the form (11) one deduces that the number of infinite Bethe roots transforms according to

$$n_\infty \to M - n_\infty - 2n_B - n_S N' \tag{54}$$

while the finite Bethe roots and complete string centres of the respective eigenvalues are related by the transformations

$$z_j^B(q) \to 1/z_j^B(q^{-1}) \qquad \text{and} \qquad z_j^S(\mu^N) \to 1/z_j^S(\mu^N). \tag{55}$$

Finally one finds for the normalization constants the mapping

$$\mathcal{N}_\mu(q) \to \mathcal{N}_{\mu^{-1}}(q^{-1})(-1)^{M+n_S N'} \mu^{-M-2n_B} q^M \prod_{j=1}^{n_B} z_j^B(q^{-1})^2 \prod_{j=1}^{n_S} z_j^S(\mu^{-1})^{N'}. \tag{56}$$

The case of real eigenvectors can only happen when the corresponding eigenvalues of the translation operator $T(z=1,q)$ are real, i.e. $k = 0, \pi$. Then (52) becomes an identity in terms of eigenvalues and Bethe roots as well as string centres must be invariant under the transformation laws (54), (55). This implies for $v(q) = v(q^{-1})$ the identities

$$M = 2n_\infty + 2n_B + n_S N' \tag{57}$$

$$P_B(z^{-1}, q^{-1}) = (-z)^{-n_B} P_B(z,q) \prod_{j=1}^{n_B} z_j^B(q)^{-1} \tag{58}$$

$$P_S(z^{-N'}, \mu^{-N}) = P_S(z^{N'}, \mu^N)(-z^{N'})^{-n_S} \prod_{j=1}^{n_S} z_j^S(\mu)^{-N'}. \tag{59}$$

Note that (57) fixes the phase factor in (51) to be $q^{-n_S N'} = (-1)^{n_S(N+1)}$ and completes the derivation of the Bethe ansatz equations (2) for odd roots of unity and real eigenvectors. For N even we will see momentarily that there is always an even number of complete strings, whence (2) also applies in this case.

The relation (57) also implies that in a degenerate eigenspace of the transfer matrix with real eigenvectors and a fixed set of Bethe roots the number of complete strings is constant. Below we will see for $N = 3$ that this also holds true for complex eigenvectors by proving the functional equation (17) and the identity (19).

Inserting the expressions (58) and (59) into the identity (52) yields the following equation for the complete string centres:

$$(\mathcal{N}_\mu/\mathcal{N}_{\mu^{-1}})\mu^{M+2n_B} \prod_{j=1}^{n_S} z_j^S(\mu)^{N'} = q^{2n_B} \prod_{j=1}^{n_B} \left(z_j^B\right)^2 = 1. \tag{60}$$

Here we have used that

$$\lim_{z\to 1} T(z,q) = (-q)^{-n_B} \prod_{j=1}^{n_B} z_j^B(q)^{-1} = \pm 1. \tag{61}$$

Exploiting that $z_j^S(\mu^{-1}q^{\pm 1})^{N'} = z_j^S(\mu^{-1})^{N'}$ and (50) leads to the further restriction

$$(\mu q)^{M+2n_B}(\mathcal{N}_{\mu q}/\mathcal{N}_{\mu^{-1}q^{-1}}) = q^{M-2n_B-2n_\infty}\mu^{M+2n_B}(\mathcal{N}_\mu/\mathcal{N}_{\mu^{-1}}) \Rightarrow q^{n_S N'} = 1. \tag{62}$$

Thus, in the case of even roots of unity there is always an even number of complete strings. This completes also the derivation of the Bethe ansatz equations (2) for even roots of unity by showing that the phase factor in (51) is equal to one.

Note that up to this point all relations have been derived for general $N' \geqslant 3$. Thus, while the conjecture (9) is only proved for $N = 3$ in this paper, the derivation of the spectrum

of the auxiliary matrices applies to all roots of unity once the commutation relations (9) are established.

6. A new functional equation for $N = 3$

In this section we prove for $q^3 = 1$ the two important formulae (19) and (18) employing the functional equation (17). While the explicit form of this functional equation is characteristic to the case $N = 3$ the representation theoretic method applied to derive it is not. Indeed, the line of argument which employs the decomposition of tensor products of evaluation representations via exact sequences also applies to the general case which we leave to future work. We only review the key steps in the derivation of (17), since the strategy is analogous to the one used in [1] to derive (3).

We start by recalling the concept of an evaluation representation. The root-of-unity representation (29) of the finite quantum algebra $U_q(sl_2)$ can be extended to a representation π^μ_w of the quantum loop algebra $U_q(\widetilde{sl}_2)$ setting

$$\begin{aligned} &\pi^\mu_w(e_0) = w\pi^\mu(f) \qquad \pi^\mu_w(f_0) = w^{-1}\pi^\mu(e) \qquad \pi^\mu_w(k_0) = \pi^\mu(t)^{-2} \\ &\pi^\mu_w(e_1) = \pi^\mu(e) \qquad \pi^\mu_w(f_1) = \pi^\mu(f) \qquad \pi^\mu_w(k_1) = \pi^\mu(t)^2. \end{aligned} \tag{63}$$

Here $\{e_i, f_i, k_i\}_{i=0,1}$ denotes the Chevalley–Serre basis of the quantum loop algebra (see e.g. [1] for further details and the conventions used). Employing the coproduct

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \qquad \Delta(k_i) = k_i \otimes k_i \tag{64}$$

one can build tensor products of representations. In the present context we consider the tensor product $\pi^\mu_w \otimes \pi^\nu_u$ of the evaluation representations associated with (29). Without loss of generality we can set $u = 1$. The corresponding representation spaces, which we denote by the same symbol, correspond to the auxiliary spaces of the Q-matrices on the left-hand side of the functional equation (17). If the evaluation parameter w is set to the special value $w = q/\mu\nu$ the tensor product $\pi^\mu_w \otimes \pi^\nu_1$ becomes decomposable, i.e. it contains subrepresentations W_1, W_2 of the quantum loop algebra giving rise to the non-split exact sequence

$$0 \to W_1 \overset{\iota}{\hookrightarrow} \pi^\mu_w \otimes \pi^\nu_1 \overset{\tau}{\to} W_2 \to 0 \qquad w = q/\mu\nu. \tag{65}$$

Here $\iota : W_1 \hookrightarrow \pi^\mu_w \otimes \pi^\nu_1$ is the inclusion and $\tau : \pi^\mu_w \otimes \pi^\nu_1 \to W_2 = \pi^\mu_w \otimes \pi^\nu_1 \big/ W_1$ the quotient projection. The representations W_1, W_2 respectively the maps ι, τ have to be explicitly calculated. This can be achieved by using the intertwiner $S(w) : \pi^\mu_w \otimes \pi^\nu_1 \to \pi^\mu_w \otimes \pi^\nu_1$ detailed in the appendix and exploiting the fact that $\ker S(q/\mu\nu) = W_1$. One finds

$$W_1 = \pi^{\mu'}_{w'} \otimes \pi^0_{z'} \qquad \text{and} \qquad W_2 = \pi^\mu_w \otimes \pi^\nu_1 \big/ W_1 = \pi^{\mu'}_{w''} \tag{66}$$

with the various parameters given by

$$w = q/\mu\nu \qquad \mu' = \mu\nu q \qquad w' = w'' = w\nu q \qquad z' = w\mu q. \tag{67}$$

Here π^0 is the root-of-unity limit, $q^3 \to 1$, of the two-dimensional representation of $U_q(sl_2)$ in terms of Pauli matrices and $\pi^0_{z'}$ the associated evaluation representation of the quantum loop algebra. The explicit form of the inclusion and quotient projection is given in the appendix. The functional equation (17) now follows from the definitions (20), (26) and the identities

$$\begin{aligned} &L^\mu_{13}(z/\mu)L^\nu_{23}(z\nu q^2)(\iota \otimes 1) = q(z-1)(\iota \otimes 1)L^{\mu\nu q}_{13}(z\mu^{-1}\nu q)R_{23}(zq) \\ &(\tau \otimes 1)L^\mu_{13}(z/\mu)L^\nu_{23}(z\nu q^2) = (zq^2-1)L^{\mu\nu q}(z\mu^{-1}\nu q)(\tau \otimes 1) \end{aligned} \tag{68}$$

which can be verified by explicit calculation using the definitions (21), (28) and the results (A.2), (A.3) in the appendix.

Expressing the functional equation (17) in terms of the eigenvalues (11) we infer that the following ratio:

$$\frac{Q_\mu(zq^2)Q_\nu(zq\nu^2)}{Q_{\mu\nu q}(zq\nu^2)} = z^{n_\infty}P_B(zq^2)P_B(zq^{-2})\frac{\mathcal{N}_\mu\mathcal{N}_\nu P_S(z^{N'},\mu^N)P_S(z^{N'}\nu^{2N'},\nu^N)}{q^{n_\infty+2n_B}\mathcal{N}_{\mu\nu q}P_S(z^{N'}\nu^{2N'},\mu^N\nu^N)} \tag{69}$$

must be independent of the parameters μ, ν. Here we have used the previous results (16) and (50). Consequently, we must have that the ratio

$$\mathcal{N}_\mu\mathcal{N}_\nu/\mathcal{N}_{\mu\nu} = \lim_{\mu\to 1}\mathcal{N}_\mu \equiv \mathcal{N} \tag{70}$$

is independent of μ, ν. Furthermore, in order that the dependence on μ, ν from the complete string contribution cancels one is led to the conclusion (18). Namely, the string centre $z_j^S(\mu)$ does either not depend on the parameter μ at all or it just depends on it via the simple factor μ^2. It follows that the ratio (69) simplifies to the expression:

$$\frac{Q_\mu(zq^2)Q_\nu(zq\nu^2)}{Q_{\mu\nu q}(zq\nu^2)} = \mathcal{N}z^{n_\infty}P_B(zq^2)P_B(zq)P_S(z^{N'},\mu^N=1).$$

Inserting this identity into (17) and using the previously derived formula (15) for the eigenvalues of the six-vertex transfer matrix completes the proof of the desired equality (19) for $N=3$.

We conclude this section by noting that the result (18) now also allows us to derive the μ-dependence of the normalization constant in (11) for real eigenvectors. Employing (60) and (70) setting $\mu\to\mu^{-1/2}$, $\nu\to\mu$ one arrives at

$$\mathcal{N}_\mu = \mathcal{N}\mu^{-\frac{M}{2}-n_B-N'n'_S} \qquad \mu^{N'n'_S} = \prod_{j=1}^{n_S}\left[z_j^S(\mu^{1/2})\big/z_j^S\right]^{N'}. \tag{71}$$

Here n'_S is simply the number of exact string centres which depend on μ^2. The above identity in particular implies

$$\mathcal{N}_{\mu q} = \mathcal{N}_\mu q^{-\frac{M}{2}-n_B-N'n'_S} \tag{72}$$

which fixes the arbitrary sign factor in (50) and thus in (15).

7. Discussion

In this paper we have analysed the eigenvalues of the auxiliary matrices (26) at roots of unity belonging to the Abelian subvariety (8). Let us now summarize what we have learnt from their spectra about the eigenvalues and eigenspaces of the six-vertex model at roots of 1.

Our starting point and motivation [1] for employing auxiliary matrices was the observation that when $q^N=1$ the more commonly used approaches such as the coordinate space [14] or algebraic Bethe ansatz [15] have serious drawbacks. For example, one derives from the algebraic Bethe ansatz away from a root of unity the following expression for the eigenvalues of the transfer matrix (20):

$$q^N\neq 1: \quad T(z,q) = b(z,q)^M q^{-n_B}\prod_{j=1}^{n_B}\frac{zq^2-z_j^B(q)}{z-z_j^B(q)} + q^{n_B}\prod_{j=1}^{n_B}\frac{zq^{-2}-z_j^B(q)}{z-z_j^B(q)} \tag{73}$$

with

$$q^N\neq 1: \quad n_B = \frac{M}{2}-|S^z|. \tag{74}$$

Here we have set as before $z_j^B = \mathrm{e}^{u_j^B} q^{-1}$. One often finds in the literature that this parametrization is used for all real coupling values even though it breaks down when $q^N = 1$. This does not mean that the root-of-unity limit $q^N \to 1$ in (73) is ill-defined, it simply requires the explicit knowledge of the Bethe roots. The latter, however, are usually not known due to the intricate nature of the Bethe ansatz equations (2). What can happen in the root-of-unity limit is that some of the Bethe roots drop out of the parametrization (73). As a concrete example consider the spin-zero sector of the $M = 6$ chain when $q^N \neq 1$. One finds the three Bethe roots

$$M = 6 \qquad S^z = 0: \quad z_1^B = 1 \qquad z_2^B = q^{-1} \qquad z_3^B = q^{-2}$$

which belong to one of the eigenvalues of the transfer matrix in the four-dimensional momentum $k = 0$ sector. If the limit $q^3 \to 1$ is taken the products over the Bethe roots in (73) give simply one and the eigenvalue becomes degenerate with the eigenvalue of the pseudo-vacuum consisting of the state where all spins point up (down).

Since the Bethe roots are not known in general one needs a parametrization of the eigenvalues in terms of the finite solutions to the Bethe ansatz equations (2) when $q^N = 1$. This parametrization we found to be

$$q^N = 1: \quad T(z) = b(z)^M q^{-\frac{M}{2}+n_\infty} \prod_{j=1}^{n_B} \frac{zq^2 - z_j^B(q)}{z - z_j^B(q)} + q^{\frac{M}{2}-n_\infty} \prod_{j=1}^{n_B} \frac{zq^{-2} - z_j^B(q)}{z - z_j^B(q)}$$

where the number of Bethe roots is not fixed by the total spin in contrast to (74). Instead, we found the sum rule

$$q^N = 1: \quad M - 2n_\infty - 2n_B = 0 \bmod N \tag{75}$$

for real eigenvectors. Numerical examples for the $M = 3, 4, 5, 6, 8$ chain and $N = 3, 4, 5, 6$ showed so far that it also extends to complex eigenvectors provided $n_B \neq 0$. The above sum rule also played an important role in the derivation of the Bethe ansatz equations (2) from the functional equation (3). This derivation involved an additional phase factor which according to (75) is trivial.

Note that the difference between $q^N \neq 1$ and $q^N = 1$ is not 'only' a difference in the number of Bethe roots and a change of the phase factors in front of the products in (73). In the root-of-unity limit also the eigenstates of the transfer matrix 're-organize' into degenerate eigenspaces across sectors of different spin. The main objective outlined in the introduction was to investigate the structure of these degenerate eigenspaces. We will now discuss how this information is encoded in the complete strings (14). Recall that the complete string contribution in the eigenvalues of the auxiliary matrices (26) is already fixed by the Bethe root content via the identities (18) and (19). So far we have only explained how these results determine the dimension of the degenerate eigenspaces. For several examples we will now explicitly see how the complete strings characterize the degenerate eigenspaces in terms of irreducible representations of the loop algebra $\widetilde{sl}_2$.

7.1. The Drinfeld polynomial and complete strings

Recall that the loop algebra $\widetilde{sl}_2$ has been established as a symmetry of the six-vertex model in the commensurate sectors where the total spin is a multiple of the order of the root of unity, i.e. $2S^z = 0 \bmod N$ [6, 7]. In order to make the connection between the spectra of the auxiliary matrices and the loop algebra we need first to introduce some facts about its representation theory [30].

There are several bases to write down the loop algebra $\widetilde{sl}_2$. The most convenient one for the present purpose is the mode basis obeying the relations :

$$\begin{aligned} &h_{m+n} = \left[x_m^+, x_n^-\right] \qquad \left[h_m, x_n^\pm\right] = \pm 2 x_{m+n}^\pm \\ &[h_m, h_n] = 0 \qquad\qquad \left[x_{m+1}^\pm, x_n^\pm\right] = \left[x_m^\pm, x_{n+1}^\pm\right]. \end{aligned} \tag{76}$$

The generators $\left\{x_m^\pm, h_m\right\}_{m\in\mathbb{Z}}$ can be successively obtained from the Chevalley–Serre basis of the quantum loop algebra $U_q^{\mathrm{res}}(\widetilde{sl}_2)$ at $q^N = 1$ via the quantum Frobenius homomorphism [30, 7] (for simplicity we only consider N odd)

$$E_1^{(N)} \to x_0^+ \qquad F_1^{(N)} \to x_0^- \qquad E_0^{(N)} \to x_1^- \qquad F_0^{(N)} \to x_{-1}^+ \qquad 2S^z/N \to h_0 \tag{77}$$

and with the action of the quantum group generators given by [6, 7][3]

$$E_1^{(N)}(q) = F_0^{(N)}(q^{-1}) = \mathfrak{R}E_0^{(N)}(q)\mathfrak{R} = \mathfrak{R}F_1^{(N)}(q^{-1})\mathfrak{R} = \sum_{1\leqslant m_1<\cdots<m_N\leqslant M} 1\otimes\cdots\otimes \underset{m_1^{\mathrm{th}}}{\sigma^+}$$
$$\otimes q^{(N-1)\sigma^z}\cdots\otimes \underset{m_2^{\mathrm{th}}}{\sigma^+}\otimes q^{(N-2)\sigma^z}\cdots q^{\sigma^z}\otimes \underset{m_N^{\mathrm{th}}}{\sigma^+}\otimes 1\cdots\otimes 1.$$

Here $\mathfrak{R}$ denotes the spin-reversal operator. As we are only considering spin-chains of finite length, all representations of the loop algebra are finite-dimensional and therefore highest weight [30]. That is, there exists a highest weight vector Ω satisfying

$$x_n^+\Omega = 0 \qquad h_n\Omega = x_n^+x_0^-\Omega = x_0^+x_n^-\Omega = \lambda_n\Omega \qquad \lambda_n\in\mathbb{C}. \tag{78}$$

All finite-dimensional irreducible highest-weight representations are isomorphic to tensor products of evaluation representations [30]. Let $\pi^s : sl_2 \to \mathrm{End}\,\mathbb{C}^{2s+1}$ denote the spin s representation of the finite subalgebra $sl_2 = \left\{x_0^\pm, h_0\right\} \subset \widetilde{sl}_2$. Then define the evaluation representation $\pi_a^s : \widetilde{sl}_2 \to \mathrm{End}\,\mathbb{C}^{2s+1}$ by setting

$$\pi_a^s\left(x_0^\pm\right) = \pi^s\left(x_0^\pm\right) \qquad \pi_a^s\left(x_{\pm 1}^\mp\right) = a^{\pm 1}\pi^s\left(x_0^\mp\right) \qquad \pi_a^s(h_0) = \pi^s(h_0) \qquad a\in\mathbb{C}. \tag{79}$$

The information which evaluation representations are contained in the highest weight representation π_Ω is conveniently encoded in the classical analogue of the Drinfeld polynomial P_Ω according to the following correspondence [30]:

$$\pi_\Omega \cong \pi_{a_1}^{s_1}\otimes\cdots\otimes\pi_{a_n}^{s_n} \quad\Leftrightarrow\quad P_\Omega(u) = \prod_{j=1}^n (1-a_ju)^{2s_j}. \tag{80}$$

Here all zeros a_j are different. The Drinfeld polynomial can be explicitly calculated from the eigenvalues λ_n of the Cartan elements h_n via the following Laurent series expansions around $u = 0$ and $u = \infty$ [30]:

$$\sum_{n=0}^\infty \lambda_n u^n = \deg P_\Omega - u\frac{P_\Omega'(u)}{P_\Omega(u)} \qquad \sum_{n=1}^\infty \lambda_{-n}u^{-n} = -u\frac{P_\Omega'(u)}{P_\Omega(u)}. \tag{81}$$

The important observation in connection with the auxiliary matrices constructed in [1] is now the following: the Drinfeld polynomials of the highest weight representations spanning the degenerate eigenspaces of the transfer matrix coincide with the complete string contributions (19) appearing in the spectrum of $Q_\mu(z)$ when we identify $u = z^N$. That is, we find up to a possible renummeration the identification

$$\dim\pi_\Omega = 2^{n_S} \qquad \text{and} \qquad a_j = \lim_{\mu\to 1} z_j^S(\mu)^{-N}. \tag{82}$$

At the moment we do not have a general proof of this assertion but we have verified it for several examples; (see the appendix). We consider one of them in detail to illustrate the interplay between the auxiliary matrices and the loop algebra $\widetilde{sl}_2$.

[3] Here we have used a different convention for the coproduct than in [6, 7]. However, one analogously proves in this case that the quantum group generators commute with the six-vertex transfer matrix in the commensurate sectors.

7.1.1. Examples for $N = 3$. Consider the spin-chain with $M = 6$ sites and the primitive root of unity $q = \exp(2\pi\mathrm{i}/3)$. Then the vector with all spins up

$$M = 2N = 6: \quad \Omega = \uparrow \otimes \uparrow \cdots \otimes \uparrow \equiv |\uparrow\uparrow \cdots \uparrow\rangle \tag{83}$$

lies in the commensurate sector $S^z = 0 \bmod N$, where the loop algebra generators are defined via (77). The corresponding eigenvalue of the transfer matrix

$$T(z,q)|_{\pi_\Omega} = 1 + b(z,q)^6$$

is four-fold degenerate with the eigenspace π_Ω spanned by

$$\pi_\Omega = \{\Omega\}_{S^z=3} \oplus \{x_1^-\Omega, x_0^-\Omega\}_{S^z=0} \oplus \{\mathfrak{R}\Omega\}_{S^z=-3}.$$

Here we have indicated via the lower indices the respective spin-sectors. All eigenvectors have zero momentum. Given the highest weight vector one can now proceed and calculate the corresponding Drinfeld polynomial. From the scalar products

$$\begin{aligned}
\lambda_0 &= \langle\Omega|x_0^+x_0^-|\Omega\rangle = \langle\Omega|E_1^{(3)}F_1^{(3)}|\Omega\rangle = \tfrac{1}{2}\dim\pi_\Omega = 2\\
\lambda_1 &= \langle\Omega|x_0^+x_1^-|\Omega\rangle = \langle\Omega|E_1^{(3)}E_0^{(3)}|\Omega\rangle = a_+ + a_- = 20\\
\lambda_2 &= \langle\Omega|x_0^+x_2^-|\Omega\rangle = \langle\Omega|\big(x_0^+x_1^-\big)^2 - \tfrac{1}{2}\big(x_0^+\big)^2\big(x_1^-\big)^2|\Omega\rangle = a_+^2 + a_-^2 = 398 \quad \ldots
\end{aligned}$$

one finds (see also [31])

$$P_\Omega(u) = (1 - a_+u)(1 - a_-u) \quad \text{with} \quad a_\pm = 10 \pm 3\sqrt{11} = (10 + 3\sqrt{11})^{\pm 1}. \tag{84}$$

Diagonalizing the auxiliary matrix $Q_\mu(z)$ in the respective spin sectors one computes the following complete string contributions in the subspace of momentum zero:

$$\begin{aligned}
S^z = +3: &\quad P_S(z^3,\mu^6) = z^6 - 20z^3\mu^3 + \mu^6 = (z^3 - a_+\mu^6)(z^3 - a_-\mu^6)\\
S^z = -3: &\quad P_S(z^3,\mu^6) = z^6\mu^6 - 20z^3\mu^3 + 1 = (z^3 - a_+)(z^3 - a_-)
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
S^z = 0: \quad P_S^\pm(z^3,\mu^6) &= z^6 - z^3(10(\mu^6+1) \pm 3\sqrt{11}(\mu^6 - 1)) + 1\\
&= (z^3 - a_\pm\mu^6)(z^3 - a_\mp).
\end{aligned}$$

Note that $\{E_0^{(3)}\Omega, F_1^{(3)}\Omega\}$ are in general not eigenvectors of the auxiliary matrix, but that the eigenvectors of $Q_\mu(z)$ are contained in the two-dimensional space spanned by them. Taking the limit $\mu \to 1$ and identifying $z^3 = u$ we recover the Drinfeld polynomial (84) from the complete strings. In this limit the auxiliary matrix becomes degenerate—as the representation underlying the definition (26) becomes reducible—and $E_0^{(3)}\Omega$, $F_1^{(3)}\Omega$ are now both eigenvectors of the auxiliary matrix. However, in general we want to keep the auxiliary matrix non-degenerate and therefore μ should be chosen different from one.

The above example also nicely confirms the picture outlined in the introduction. According to (18) there are $2^{n_S} = 4$ possible eigenvalues of the auxiliary matrix in the degenerate eigenspace of the transfer matrix, all of which we find realized.

Note that the match between the complete string centres and the evaluation parameters is a virtue particular to the auxiliary matrices (26) constructed in [1]. In the context of the six-vertex model the other explicit expression in the literature is Baxter's formula (101) in [10] (which applies to all coupling values $\gamma \in \mathbb{R}$ but only to the sectors of vanishing total spin),

$$S^z = 0: \quad Q_{\mathrm{Baxter}}(z)^{\beta_1\ldots\beta_M}_{\alpha_1\ldots\alpha_M} = \exp\left(\frac{1}{4}\mathrm{i}\gamma\sum_{m=1}^{M}\sum_{n=1}^{m-1}(\alpha_m\beta_n - \alpha_n\beta_m) + \frac{1}{4}u\sum_{m=1}^{M}\alpha_m\beta_m\right). \tag{86}$$

Here $z = \mathrm{e}^u q^{-1}, q = \mathrm{e}^{\mathrm{i}\gamma}$. Diagonalizing this matrix in the two-dimensional subspace $\{E_0^{(3)}\Omega, F_1^{(3)}\Omega\}$ of the spin-zero sector we find for each of the two eigenvalues only a single

complete string with string centres $z^S = \pm 1$. Thus, for Baxter's auxiliary matrix neither the degree of the complete string contribution nor the values of the string centres are in agreement with the data obtained from the loop algebra.

Admittedly, the above example for the $M = 6$ chain is quite simple and we chose it to illustrate the working of the formulae. One might wonder if the identification of complete strings and the Drinfeld polynomial also applies when the highest weight vector is a real Bethe eigenstate, i.e. when finite Bethe roots are present. We have explicitly worked out the following examples with $q = \exp(2\pi \mathrm{i}/3)$: for the $M = 5$ chain one finds five doublets in the $S^z = \pm 3/2$ sectors and for the $M = 8$ chain there are eight quartets in the $S^z = 3, 0, -3$ sectors. In all of these cases there is an agreement between the complete strings (14) calculated from the auxiliary matrices (26) and the Drinfeld polynomial (80). The results are presented in the appendix. They also show the working of the identity (19) which yields expressions for the evaluation parameters of the loop algebra in terms of Bethe roots. Also the Bethe ansatz equations are recovered by making a Laurent series expansion in (19). This shows an intimate link between the Bethe ansatz and the representation theory of the loop algebra.

7.2. Comparison with the eight-vertex model

We conclude by mentioning that the identity (19) coincides with the trigonometric limit of a recent conjecture by Fabricius and McCoy [32] on the eigenvalues of Baxter's eight-vertex auxiliary matrix constructed in [9]. (Note that this auxiliary matrix is different from the one discussed in [10–13].) Based on numerical results for the $M = 8$ chain and $N = 3, 4, 6$ they arrive at the elliptic analogue of the identity (19) by using a functional equation similar in nature to (17) (cf equations (3.12) and (3.10) in [32]). However, there are some key differences as Baxter's construction procedure and the one used in [1] are not the same. In particular, the dependence on continuous parameters analogous to (4) respectively (8) is absent in Baxter's matrix [9]. In the context of the auxiliary matrices (26) we saw that these parameters play a crucial role in breaking the loop algebra symmetry and the invariance under spin-reversal. A direct comparison between the auxiliary matrices for the eight [9–12] and six-vertex model [1] is not straightforward: taking the trigonometric limit in the eight-vertex eigenvalues requires the knowledge of the explicit dependence of various normalization constants and the Bethe roots on the elliptic nome. Further, investigations are needed to clarify this point.

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Appendix A. The intertwiner for $\pi^{\mu}_{w} \otimes \pi^{\nu}_{1}$ with $N = 3$

In this section we construct the intertwiner for the following tensor product of evaluation representations $\pi^{\mu}_{w} \otimes \pi^{\nu}_{1}$ with $N = 3$. If this intertwiner exists the auxiliary matrices $Q_{\mu}(w)$, $Q_{\nu}(w')$ must commute, i.e. the conjecture (9) holds true for $N = 3$. The defining equation of the intertwiner S is given by

$$S(w)\big(\pi^{\mu}_{w} \otimes \pi^{\nu}_{1}\big)\Delta(x) = \big[\big(\pi^{\mu}_{w} \otimes \pi^{\nu}_{1}\big)\Delta^{\mathrm{op}}(x)\big]S(w) \qquad x \in U_q(\widetilde{sl}_2). \quad \text{(A.1)}$$

Here π_w^μ is the evaluation representation (63) obtained from (29) for $N = 3$. The symbols $\Delta, \Delta^{\mathrm{op}}$ denote the coproduct (64) and the opposite coproduct. The latter is obtained by permuting the two factors in (64). The defining equation (A.1) yields a system of algebraic equations for the matrix elements of the intertwiner. As S commutes with $\left(\pi_w^\mu \otimes \pi_1^\nu\right)\Delta(k_i)$ it is convenient to decompose the tensor product space into the following direct sum:

$$V = V_1 \oplus V_2 \oplus V_3$$

where the respective subspaces are spanned by the following basis vectors:

$$\begin{aligned} V_1 &= \mathrm{span}\{v_0 \otimes v_0,\, v_1 \otimes v_2,\, v_2 \otimes v_1\} \\ V_2 &= \mathrm{span}\{v_0 \otimes v_1,\, v_1 \otimes v_0,\, v_2 \otimes v_2\} \\ V_3 &= \mathrm{span}\{v_0 \otimes v_2,\, v_1 \otimes v_1,\, v_2 \otimes v_0\}. \end{aligned}$$

Here $v_i, i = 0, 1, 2$ denotes the standard basis in $\mathbb{C}^3$ used in the definition (29) of the representation π^μ. The calculation is cumbersome but straightforward and one finds the following solution up to a common normalization factor:

$$S|_{V_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{q(w\mu-\nu)(w\mu\nu-q)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{(w\mu\nu-q)(\mu^2-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} \\ 0 & \frac{w(w\mu\nu-q^2)(\nu^2-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{q(w\nu-\mu)(w\mu\nu-q)}{(wq-\mu\nu)(wq^2-\mu\nu)} \end{pmatrix}$$

$$S|_{V_2} = \begin{pmatrix} \frac{q^2(w\mu-\nu)}{wq-\mu\nu} & \frac{q-\mu^2}{wq-\mu\nu} & 0 \\ \frac{w(q-\nu^2)}{wq-\mu\nu} & \frac{q^2(w\nu-\mu)}{wq-\mu\nu} & 0 \\ 0 & 0 & \frac{(w\mu\nu-q)(w\mu\nu-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} \end{pmatrix}$$

$$S|_{V_3} = \begin{pmatrix} \frac{q^2(w\mu-\nu)(w\mu-\nu q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{(q-\mu^2)(w\mu-\nu)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{(\mu^2-q)(\mu^2-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} \\ \frac{w(w\mu-\nu)(\nu^2-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{\mu\nu(1+q^2w^2)+wq(\mu^2+1)(\nu^2+1)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{(\mu^2-q^2)(w\nu-\mu)}{(wq-\mu\nu)(wq^2-\mu\nu)} \\ \frac{w^2(\nu^2-q)(\nu^2-q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{w(q-\nu^2)(w\nu-\mu)}{(wq-\mu\nu)(wq^2-\mu\nu)} & \frac{q^2(w\nu-\mu)(w\nu-\mu q^2)}{(wq-\mu\nu)(wq^2-\mu\nu)} \end{pmatrix}.$$

We can now use this solution in order to explore the decomposition of the tensor product at special values of the evaluation parameter w. One can explicitly verify that for $w = q/\mu\nu$ the intertwiner has a non-trivial kernel consisting of the following six-dimensional space:

$$\begin{aligned} \ker S_1(q/\mu\nu) &= \mathrm{span}\{v_1 \otimes v_2,\, v_2 \otimes v_1\} \\ \ker S_2(q/\mu\nu) &= \mathrm{span}\left\{v_2 \otimes v_2,\, \frac{(\mu^2-q)\nu}{1-q^2\nu^2} v_0 \otimes v_1 + v_1 \otimes v_0\right\} \\ \ker S_3(q/\mu\nu) &= \mathrm{span}\left\{\frac{\nu^2(\mu^2-q)(\mu^2-q^2)}{(q-\nu^2)(\nu^2 q-1)} v_0 \otimes v_2 \right. \\ &\quad \left. + v_2 \otimes v_0,\, \frac{\nu(q-\mu^2)}{\nu^2 q-1} v_0 \otimes v_2 + v_1 \otimes v_1\right\}. \end{aligned}$$

This kernel can be identified as a submodule $W_1 \subset \pi_w^\mu \otimes \pi_1^\nu$ of the quantum loop algebra.

Appendix A.1. The inclusion $\iota : \pi_{w'}^{\mu'} \otimes \pi_{z'}^0 \hookrightarrow W_1 \subset \pi_w^\mu \otimes \pi_1^\nu$

We define the module W_1 simply by stating the inclusion of the basis vectors spanning the tensor product $\pi_{w'}^{\mu'} \otimes \pi_{z'}^0$ into the tensor product $\pi_w^\mu \otimes \pi_1^\nu$. Denote by $\{v_i'\}$ the basis vectors in $\pi_{w'}^{\mu'}$ and by $\{\uparrow, \downarrow\}$ the basis vector of the two-dimensional representation of $U_q(sl_2)$. Then

the inclusion ι is defined by linear extension from the following relations involving the basis vectors:

$$\begin{aligned}
&\iota(v_2'\otimes\downarrow)=\alpha v_2\otimes v_2 \qquad \iota(v_2'\otimes\uparrow)=\alpha v_2\otimes v_1\\
&\iota(v_1'\otimes\uparrow)=\gamma_0\{\gamma_1 v_1\otimes v_1+v_2\otimes v_0\} \qquad \iota(v_1'\otimes\downarrow)=\gamma_2 v_1\otimes v_2+\gamma_3 v_2\otimes v_1\\
&\iota(v_0'\otimes\uparrow)=\beta_0 v_0\otimes v_1+v_1\otimes v_0 \qquad \iota(v_0'\otimes\downarrow)=\beta_0 v_0\otimes v_2+\beta_1 v_1\otimes v_1+\beta_2 v_2\otimes v_0.
\end{aligned} \tag{A.2}$$

The coefficients in the above linear combinations are given by

$$\begin{aligned}
\alpha &= \frac{(\mu^2\nu^2-1)(1-q\mu^2\nu^2)}{\nu(\mu^2 q-1)(\nu^2-q)}\\
\beta_0 &= \frac{(\mu^2-q)\nu}{1-q^2\nu^2} \qquad \beta_1=\nu\beta_0+1-\gamma_0\gamma_1 q^2 \qquad \beta_2=\nu q-\gamma_0 q^2\\
\gamma_0 &= \frac{\alpha(1-\nu^2 q^2)}{\mu^2\nu^2-1} \qquad \gamma_1=\frac{\nu(\mu^2-q^2)}{q-\nu^2} \qquad \gamma_2=(\beta_0 q\nu-1)\\
\gamma_3 &= [q\alpha+\nu(q^2-\nu\beta_0).
\end{aligned}$$

Acting with the quantum group generators according to (64) on the basis vectors in the respective tensor products of evaluation representations one verifies that the above inclusion is well-defined.

A.2. The projection $\tau:\pi_w^\mu\otimes\pi_1^\nu\to W_2=\pi_w^\mu\otimes\pi_1^\nu\big/W_1$

Having identified the submodule W_1 it remains to verify that the quotient space W_2 defines the evaluation representation $\pi_{w''}^{\mu''}$ as outlined in (65), (66) and (67). This follows when identifying the equivalence classes of the following vectors in $\pi_w^\mu\otimes\pi_1^\nu$ with the basis vectors $\{v_i''\}$ in $\pi_{w''}^{\mu''}$:

$$\begin{aligned}
v_0'' &\equiv \tau(v_0\otimes v_0)\\
v_1'' &\equiv \tau(v_0\otimes v_1+\nu q v_1\otimes v_0)\\
v_2'' &\equiv \tau(v_0\otimes v_2-\nu q^2 v_1\otimes v_1+\nu^2 q^2 v_2\otimes v_0).
\end{aligned} \tag{A.3}$$

This concludes the proof of the decomposition (65). Using the explicit form of the inclusion and projection map one is now in the position to proof the functional equation (17) as described in the text.

Appendix B. Calculation of the Drinfeld polynomial

We present several examples of calculating the evaluation parameters (80) of the loop algebra in the degenerate eigenspaces of the transfer matrix when $q=\exp(2\pi\mathrm{i}/3)$. We then compare the outcome with the expression (19) derived from the complete strings of the auxiliary matrices.

B.1. $M=5$, $S^z=\pm 3/2$

There are in total five doublets for the $M=5$ chain in the sectors $S^z=\pm 3/2$. The corresponding highest weight vectors Ω_k can be labelled by their momenta and are defined as follows:

$$\Omega_k=\sum_{n=1}^{5}\mathrm{e}^{\mathrm{i}nk}T(1,q)^n\,|\uparrow\uparrow\uparrow\uparrow\downarrow\rangle \qquad k/\pi=0,\pm 2/5,\pm 4/5.$$

Since there are only doublets occurring in this example the corresponding Drinfeld polynomials P_{Ω_k} defined in (80) contain only one factor. For each highest weight vector the corresponding evaluation parameter $a(k)$ is calculated using the action of the loop algebra,

$$x_0^+ x_1^- \Omega_k = [4 + 3q\,\mathrm{e}^{-ik} + (1+2q^2)\,\mathrm{e}^{-2ik} + (1+2q)\,\mathrm{e}^{-3ik} + 3q^2\,\mathrm{e}^{-4ik}]\Omega_k = a(k)\Omega_k. \tag{B.1}$$

In order to compare this result with the complete strings we may either directly diagonalize the auxiliary matrices (26) in the respective spin-sectors or use the identity (19). In the latter approach one first solves the Bethe ansatz equation

$$1 = q^5 \left(\frac{1 - z_B q^2}{1 - z_B}\right)^5 \tag{B.2}$$

and then computes from the Bethe roots the complete strings in the limit $\mu \to 1$,

$$\mathcal{N}\big(z^3 - z_S^3\big) = \sum_{\ell \in \mathbb{Z}_3} \frac{(zq^{2\ell} - 1)^5}{(zq^{2\ell} - z_B)(zq^{2(\ell+2)} - z_B)} = -\frac{3}{z_B^2}\left(1 + \frac{1 - 10z_B^2(z_B + q^2)}{z_B^3} z^3\right). \tag{B.3}$$

Bethe roots and momenta can be easily matched by taking the limit $z \to 1$ in (15) yielding the following second identity for the evaluation parameter:

$$a(k) = 10 + 10q^2/z_B - 1\big/z_B^3 \qquad \mathrm{e}^{ik} = -q^{\frac{1}{2}} \frac{1 - z_B q^2}{1 - z_B}. \tag{B.4}$$

Note that the Bethe ansatz equations (B.2) are recovered from the Laurent series expansion in (B.3) by setting all coefficients of the terms with powers greater than three equal to zero.

B.2. $M = 8$, $S^z = 3, 0, -3$

For the $M = 8$ chain one proceeds similar as in the previous case. One now has eight quartets whose highest weight vectors in the $S^z = 3$ sector are again labelled by their momenta

$$\Omega_k = \sum_{n=1}^{8} \mathrm{e}^{ink} T(1,q)^n |{\downarrow}{\uparrow}{\uparrow}{\uparrow}{\uparrow}{\uparrow}{\uparrow}{\uparrow}\rangle.$$

The degree of Drinfeld polynomial P_{Ω_k} is now two, i.e. there are two evaluation parameters $a_\pm = a_\pm(k)$ to compute. After some cumbersome computations one obtains

$$\begin{aligned} a_+ + a_- = \langle \Omega_k | x_0^+ x_1^- | \Omega_k \rangle &= 35 + 15q\,\mathrm{e}^{-ik} + 5i\sqrt{3}q^2\,\mathrm{e}^{-2ik} - (5 - 2q)\,\mathrm{e}^{-3ik} \\ &\quad + 6\,\mathrm{e}^{-4ik} - (5 - 2q^2)\,\mathrm{e}^{-5ik} - 5i\sqrt{3}q\,\mathrm{e}^{-6ik} + 15q^2\,\mathrm{e}^{-7ik} \end{aligned}$$

and

$$\begin{aligned} 4a_+a_- &= \langle \Omega_k | \big(x_0^+\big)^2 \big(x_1^-\big)^2 | \Omega_k \rangle \\ &= 4(\mathrm{e}^{-ik} + q\,\mathrm{e}^{-2ik} + q^2\,\mathrm{e}^{-3ik} + \mathrm{e}^{-4ik} + q\,\mathrm{e}^{-5ik} + q^2\,\mathrm{e}^{-6ik} + \mathrm{e}^{-7ik})^2. \end{aligned}$$

Again we can compare this result against the complete string by diagonalizing the auxiliary matrices or by employing the identity (19). In either case one finds upon matching string centres and evaluation parameters the following expression in terms of the Bethe roots:

$$a_+ + a_- = 56 + 28q^2/z_B - 1\big/z_B^3 \qquad a_+a_- = 28 + 56q^2/z_B - 56\big/z_B^3 - 28q^2\big/z_B^4 + 1\big/z_B^6.$$

Here Bethe roots and momenta are related by the identity

$$\mathrm{e}^{ik} = q^2 \frac{1 - z_B q^2}{1 - z_B}.$$

In order to facilitate the comparison we have summarized the results in the table below.

Momentum	String centres/evaluation parameters $a_\pm$	Bethe root z^B/q^2
$k=0$	$\frac{1}{2}(29\pm 3\sqrt{93})$	-1
$k=\pi$	$\frac{1}{2}(83\pm 9\sqrt{85})$	1
$k=\pi/2$	$\frac{1}{2}\left(13(2+\sqrt{3})\pm\sqrt{165(7+4\sqrt{3})}\right)$	$-2-\sqrt{3}$
$k=-\pi/2$	$\frac{1}{2}\left(13(2-\sqrt{3})\pm\sqrt{165(7-4\sqrt{3})}\right)$	$-2+\sqrt{3}$
$k=\pi/4$	$a_+=38.971\ldots$ $a_-=0.068\,0614\ldots$	$1-\frac{3}{\sqrt{2}}-\sqrt{\frac{3}{2}(2-2\sqrt{2})}$
$k=3\pi/4$	$a_+=59.9864\ldots$ $a_-=0.615\,865\ldots$	$1+\frac{3}{\sqrt{2}}+\sqrt{\frac{3}{2}(2+2\sqrt{2})}$
$k=5\pi/4$	$a_+=1.62373\ldots$ $a_-=0.016\,6705\ldots$	$1+\frac{3}{\sqrt{2}}-\sqrt{\frac{3}{2}(2+2\sqrt{2})}$
$k=7\pi/4$	$a_+=14.6926\ldots$ $a_-=0.025\,6601\ldots$	$1-\frac{3}{\sqrt{2}}+\sqrt{\frac{3}{2}(2-2\sqrt{2})}$

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